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On a Conjecture of F. Móricz and X. L. Shi

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Let $f(x, y)$ be a continuous function, 2π-periodic in each variable, in symbols $f \in C_{2\pi \times 2\pi}$. The paritial moduli of continuity of f are defined for $\delta > 0$ by

$$\omega_x(f, \delta) = \sup_{|u| \leqslant \delta} \max_{x, y} |f(x+u, y) - f(x, y)|,$$

and

$$\omega_y(f, \delta) = \sup_{|v| \leqslant \delta} \max_{x, y} |f(x, y+v) - f(x, y)|.$$

Moreover, the Lipschitz class $\mathrm{Lip}(\alpha, \beta)$, where $\alpha, \beta \in (0, 1]$, is defined to be

$$\mathrm{Lip}(\alpha, \beta) = \{f \in C_{2\pi \times 2\pi} : \omega_x(f, \delta) = O(\delta^\alpha) \quad \text{and} \quad \omega_y(f, \delta) = O(\delta^\beta)\}.$$

The corresponding conjugate function of $f(x)$ is

$$\tilde{f}^{(1,1)}(x, y) = \frac{1}{\pi^2} \int_0^\pi \frac{1}{2} \cot \frac{v}{2}\, dv \int_0^\pi (f(x+u, y+v) - f(x-u, y+v) - f(x+u, y-v) + f(x-u, y-v)) \frac{1}{2} \cot \frac{u}{2}\, du.$$

Concerning the approximation to continuous functions by Cesàro means of double conjugate series, F. Móricz and X. L. Shi [1] raised two conjectures, one of which is as follows.

Conjecture MS [1, p. 360, Conjecture 2]. There exists a function $f \in \mathrm{Lip}(1, 1)$ such that the estimate

$$\omega_x(\tilde{f}^{(1,1)}, \delta) = o\left(\delta \ln^2 \frac{1}{\delta}\right) \quad (\delta \to +0)$$

cannot hold.

We now show that the answer to Conjecture MS is affirmative.

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THEOREM. *There exists a function* $f \in \mathrm{Lip}(1, 1)$ *such that*

$$\overline{\lim_{\delta \to +0}}\ \omega_x(f^{(1,1)}, \delta)\Big/\left(\delta \ln^2 \frac{1}{\delta}\right) > 0.$$

The following lemmas are needed.

LEMMA 1. *If* $|x| \geqslant n^{-1}$ *or* $|x| \leqslant n^{-4}$, *then the following estimate holds*:

$$\left|\sum_{k=n^2+1}^{n^3} \frac{\sin kx}{k}\right| = O\left(\frac{1}{n}\right).$$

Proof. Evidently

$$\left|\sum_{k=n^2+1}^{n^3} \frac{\sin kx}{k}\right| \leqslant |x| n^3, \tag{1}$$

on the other hand, by Abel transform

$$\sum_{k=n^2+1}^{n^3} \frac{\sin kx}{k} = \sum_{k=n^2+1}^{n^3-1} \left(\frac{1}{k} - \frac{1}{k+1}\right) S_k(x) - \frac{1}{n^2+1} S_{n^2}(x) + \frac{1}{n^3} S_{n^3}(x),$$

where

$$S_k(x) = \sum_{j=1}^{k} \sin jx = \frac{\cos x/2 - \cos (k+1/2)x}{2 \sin x/2},$$

so

$$\sum_{k=n^2+1}^{n^3} \frac{\sin kx}{k^2} \leqslant \frac{\pi}{x} \frac{2}{n^2}, \tag{2}$$

in the case $|x| \leqslant n^{-4}$ or $|x| \geqslant n^{-1}$, applying (1) or (2), it follows that

$$\left|\sum_{k=n^2+1}^{n^3} \frac{\sin kx}{k}\right| = O(n^{-1}).$$

LEMMA 2. *Let* $m = n^3$,

$$h_m(x, y) = \sum_{k=n^2+1}^{n^3} \frac{\cos kx}{k^2} \sum_{j=1}^{n} \frac{\sin jy}{j},$$

then

$$|h_m(x+\delta, y) - h_m(x, y)| = O\left(\left|\sum_{k=n^2+1}^{n^3} \frac{\sin kx_0}{k}\right| \delta\right), \qquad x_0 \in [x, x+\delta], \tag{3}$$

$$\omega_x(h_m, \delta) = O(n^{-1} \delta), \tag{4}$$

and

$$C_1\,\delta\ln^2(m+1)\leqslant\omega_x(\bar{h}_m^{(1,1)},\delta)\leqslant C_2\,\delta\ln^2(m+1),\qquad 0<\delta\leqslant m^{-1},\tag{5}$$

where C_i $(i=1,2)$ are positive constants independent of m.

Proof. Let

$$\begin{aligned}|h_m(x+\delta,y)-h_m(x,y)|\\ \leqslant\delta\left|\sum_{k=n^2+1}^{n^3}\frac{\sin k(x+\theta_1\delta)}{k}\right|\left\|\sum_{j=1}^{n}\frac{\sin jy}{j}\right\|,\qquad 0\leqslant\theta_1\leqslant 1,\end{aligned}$$

in view of $\sup_{n\geqslant 1}\|\sum_{k=1}^{n}\sin jx/j\|\leqslant 3\sqrt{\pi}$, (3) is valid. And

$$|h_m(x,y+\delta)-h_m(x,y)|=O\left(\delta\sum_{k=n^2}^{\infty}\frac{1}{k^2}\left\|\sum_{k=1}^{n}\cos j(y+\theta_2\delta)\right\|\right)=O(\delta n^{-1}),$$

where $0\leqslant\theta_2\leqslant 1$, (4) is proved. At last, it is not difficult to see

$$\bar{h}_m^{(1,1)}(x,y)=-\sum_{k=n^2+1}^{n^3}\frac{\sin kx}{k^2}\sum_{j=1}^{n}\frac{\cos jy}{j},$$

so that there is a $\theta_3\in[0,1]$ such that for $\delta\in(0,m^{-1})$

$$\begin{aligned}|\bar{h}_m^{(1,1)}(\delta,0)-\bar{h}_m^{(1,1)}(0,0)|&=\sum_{k=n^2+1}^{n^3}\frac{\cos k(\theta_3\delta)}{k}\sum_{j=1}^{n}\frac{1}{j}\\ &\geqslant\delta\cos 1(\ln(n^3+1)-\ln n^2-1)\ln(n+1)\\ &\geqslant C\,\delta\ln^2(n+1)\geqslant C_1\,\delta\ln^2(m+1),\end{aligned}$$

the converse inequality is evident, thus (5) follows.

Proof of Theorem. Let $n_l=2^{4^{l^2}}$,

$$f(x,y)=\sum_{l=0}^{\infty}h_{n_l}(x,y),$$

then

$$\bar{f}^{(1,1)}(x,y)=\sum_{l=0}^{\infty}\bar{h}_{n_l}^{(1,1)}(x,y).$$

Now set $n_{k+1}^{-1}<\delta\leqslant n_k^{-1}$,

$$|f(x+\delta, y)-f(x, y)| \leqslant \left|\sum_{l=0}^{k-1}(h_{n_l}(x+\delta, y)-h_{n_l}(x, y))\right|+\omega_x(h_{n_k}, \delta)$$

$$+\omega_x(h_{n_{k+1}}, \delta)+\sum_{l=k+2}^{\infty} \omega_x(h_{n_l}, \delta)$$

$$=I_1+I_2+I_3+I_4,$$

$$I_4 \leqslant \sum_{l=k+2}^{\infty} \sum_{i=n_l^{2/3}+1}^{n_l} \frac{1}{i^2}\left|\sum_{j=1}^{n_l^{1/3}} \frac{\sin jx}{j}\right| \leqslant 3\sqrt{\pi}\, Cn_{k+2}^{-2/3}=o(n_{k+1}^{-1})=o(\delta), \tag{6}$$

by (3)

$$I_2=O(\delta), \tag{7}$$

$$I_3=O(\delta). \tag{8}$$

From the mean value theorem for $1 \leqslant l \leqslant k-1$

$$\left|\sum_{l=0}^{k-1}(h_{n_l}(x+\delta, y)-h_{n_l}(x, y))\right|$$

$$\leqslant C \sum_{l=0}^{k-1}\left|\sum_{k=n_l^{2/3}+1}^{n_l} \frac{\sin kx_0}{k}\right| \delta, \qquad x_0 \in [x, x+\delta],$$

Note that $(n_i^{-4/3}, n_i^{-1/2}) \cap (n_j^{-4/3}, n_j^{-1/3})=\phi$, $i \neq j$, if x_0 belongs to some $(n_{l_0}^{-4/3}, n_{l_0}^{-1/3})$, then

$$I_1=\left|h_{n_{l_0}}(x+\delta, y)-h_{n_{l_0}}(x, y)+\sum_{\substack{l=1\\ l \neq l_0}}^{k-1}(h_{n_l}(x+\delta, y)-h_{n_l}(x, y))\right|,$$

applying Lemma 1, we get

$$I_1 \leqslant\left|\sum_{i=n_l^{2/3}+1}^{n_{l_0}} \frac{\sin ix_0}{i}\right| \delta+\delta \sum_{l=1}^{\infty} n_l^{-1/3}=O(\delta). \tag{9}$$

Combining (6)–(9), we have

$$\omega_x(f, \delta)=O(\delta). \tag{10}$$

It is easy to get the estimate for $\omega_y(f, \delta)$. By (4)

$$\omega_y(f, \delta)=O\left(\delta \sum_{l=1}^{\infty} n_l^{-1/3}\right)=O(\delta). \tag{11}$$

Equations (10) and (11) imply that $f(x, y) \in \mathrm{Lip}(1, 1)$. Meanwhile

$$\omega_x(\bar{f}^{(1,1)}, n_k^{-1}) \geqslant \omega_x(\bar{h}_{n_k}^{(1,1)}, n_k^{-1}) - \sum_{l=1}^{k-1} \omega_x(\bar{h}_{n_l}^{(1,1)}, n_k^{-1})$$

$$- 2 \sum_{l=k+1}^{\infty} \sum_{i=n_l^{2/3}+1}^{n_l} \frac{1}{i^2} \sum_{j=1}^{n_l^{1/3}} \frac{1}{j} = J_1 - J_2 - J_3.$$

Because of (5),

$$J_1 \geqslant C_1 n_k^{-1} \ln^2 n_k,$$

$$J_2 \leqslant C_2 n_k^{-1} \ln^2 n_k \sum_{l=1}^{k-1} \frac{\ln^2 n_l}{\ln^2 n_k} \leqslant C_2 n_k^{-1} \ln^2 n_k \sum_{l=1}^{k-1} 4^{-4kl+l^2}$$

$$= O(n_k^{-1} \ln^2 n_k 4^{-3k}),$$

$$J_3 \leqslant C \sum_{l=k+1}^{\infty} n_l^{-2/3} \ln(n_l + 1) = O(n_{k+1}^{-2/3} \ln(n_{k+1} + 1))$$

$$= o(n_k^{-1} \ln^2 n_k), \qquad k \to \infty,$$

altogether there exists a positive constant M for sufficiently large k such that

$$\omega_x(\bar{f}^{(1,1)}, n_k^{-1}) \geqslant M n_k^{-1} \ln^2 n_k,$$

i.e.,

$$\overline{\lim_{n \to \infty}}\ \omega_x(\bar{f}^{(1,1)}, n^{-1})/(n^{-1} \ln^2 n) > 0.$$

References

1. F. Móricz and X. L. Shi, Approximation to continuous functions by Cesàro means of double Fourier series and conjugate series, *J. Approx. Theory* **49** (1987), 346–377.
2. A. Zygmund, "Trigonometric Series," Cambridge Univ. Press, Cambridge, 1959.